MATH 2040 Lecture 17 (7/11/2016)

§ Schur's Lemma & Spectral Theorems

Spectral Thm: $(V, \langle , \rangle)$ inner product space, $\dim V < +\infty$.

$$T : V \to V \quad \text{linear operator}$$

$\exists$ orthonormal eigenbasis of T

$\Longleftrightarrow$ $\begin{cases} \text{(i)} \ \mathbb{F} = \mathbb{R}, \ T \ \text{self-adjoint (i.e. } T^* = T) \\ \text{(ii)} \ \mathbb{F} = \mathbb{C}, \ T \ \text{normal (i.e. } T^*T = TT^*) \end{cases}$

Recall: Some general properties of normal / self-adjoint operators.

Remember: normal $\overset{\Longleftarrow}{\not\Longrightarrow}$ self-adjoint

Lemma: Assume T normal. Then

(1) $\|Tx\| = \|T^*x\| \quad \forall x \in V$

(2) $T - cI$ normal $\quad \forall c \in \mathbb{F}$

(3) $Tx = \lambda x \Rightarrow T^*x = \bar{\lambda} x$

for some x

(4) $\left.\begin{array}{l} x_1 \in E_{\lambda_1}(T) \\ x_2 \in E_{\lambda_2}(T) \end{array} \text{ where } \lambda_1 \neq \lambda_2 \right\} \Rightarrow \langle x_1, x_2 \rangle = 0$

If T self-adjoint, then

(5) all eigenvalues of T are real.

Proof: (1) – (3) Last time.

(4): $Tx_1 = \lambda_1 x_1$, $Tx_2 = \lambda_2 x_2$

Consider

$$\lambda_1 \langle x_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle = \langle T x_1, x_2 \rangle$$

$$= \langle x_1, T^* x_2 \rangle \overset{(3)}{=} \langle x_1, \overline{\lambda_2} x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$$

$$\Rightarrow \quad \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle x_1, x_2 \rangle = 0 \qquad \Rightarrow \quad \langle x_1, x_2 \rangle = 0.$$

(5): $T = T^*$. If $\lambda \in \mathbb{C}$ is an eigenvalue for T with eigenvector $x \neq \vec{0}$, then

$$\lambda x = Tx = T^* x = \bar{\lambda} x \quad \overset{x \neq \vec{0}}{\Longrightarrow} \quad \lambda = \bar{\lambda}$$

$\blacksquare$

To prove Spectral Thm, we need the following:

Schur's Lemma: $T : V \longrightarrow V$ on an inner prod. space $(V, \langle , \rangle)$ $(\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$ $\dim V < +\infty$

Assume: The char. poly $f(t)$ of T splits over $\mathbb{F}$.

Then $\exists$ O.N.B. β for V s.t.

$$[T]_\beta = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$$

upper triangular.

Lemma: T has eigenvalue $\lambda \Rightarrow T^*$ has eigenvalue $\bar{\lambda}$

[Caution: In contrast to (3) above, they may not have a common eigenvector x.]

Proof: Fix any O.N.B. β, then $[T^*]_\beta = [T]_\beta^*$.

λ e.value for $T \Rightarrow \det(T - \lambda I) = \det([T]_\beta - \lambda I) = 0$

Check:
$$\det(T^* - \bar{\lambda} I) = \det([T^*]_\beta - \bar{\lambda} I)$$
$$= \det([T]_\beta^* - \bar{\lambda} I)$$
$$= \det\left(([T]_\beta - \lambda I)^*\right)$$
$$= \overline{\det([T]_\beta - \lambda I)} = 0$$

$$\underset{\substack{\uparrow \\ e_1 \text{ e-vector}}}{}\quad A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda = 1$$

e_1 not e-vector

$$A^t = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

e_3 e-vector.

$\lambda = 1$

eigenvectors change!

Proof of Schur's Lemma :

$$[T]_\beta = \begin{array}{cccc} v_1 & \cdots & v_{n-1} & \perp v_n \\ \end{array}$$
$$\left(\begin{array}{cccc} d_1 & & & * \\ & d_2 & & * \\ & & \ddots & * \\ 0 & \cdots & 0 & d_n \end{array}\right)$$

$\{v_1, \dots, v_{n-1}\}$ T-invariant

not restricted

- induction
- some T-invar. subspace
- choose v_n "carefully"

previous e.g. $\Rightarrow$ v_n e.vector for T^*

By induction on $\dim V = n$.

$\underline{n=1}$: trivial

Assume $n = k-1$ is true. Want true for $n = k$.

Hypothesis that $f(t)$ splits over $\mathbb{F}$

$\Rightarrow \exists$ one eigenvalue $\lambda \in \mathbb{F}$ for T

Lemma $\Rightarrow \exists$ one eigenvalue $\bar{\lambda} \in \mathbb{F}$ for T^*

$\Rightarrow \exists$ eigenvector z $(= v_n)$ for T^*

Consider $W := (\operatorname{span}\{z\})^\perp$ $k-1$ dim'l.

Claim: W is T-invariant. i.e. $T(W) \subseteq W$

Pf: Take any $w \in W$, i.e. $\langle w, z \rangle = 0$

$$\langle \underset{\in W}{Tw}, z \rangle = \langle w, T^* z \rangle = \langle w, \bar{\lambda} z \rangle = \lambda \langle w, z \rangle = 0$$

So, $T|_W : W \to W$

Q: char poly of $T|_W$ splits ? ✓

O.K. $\underbrace{f_{T_W}(t)}_{\text{splits}} \mid \underbrace{f_T(t)}_{\text{splits}}$ $\Leftarrow$

By induction, $\exists\ \beta' = \{v_1, \dots, v_{k-1}\}$ O.N.B. for W s.t.

$$[T|_W]_{\beta'} = \underbrace{\begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}}_{\text{upper triangular}}$$

Take $\beta = \beta' \cup \{z\}$, then

$$[T]_\beta = \begin{pmatrix} [T|_W]_{\beta'} & \begin{matrix} * \\ * \\ \vdots \\ * \end{matrix} \\ \underbrace{0 \cdots\cdots 0}_{W\ T\text{-inv.}} & * \end{pmatrix} \quad \therefore \text{ upper triangular}$$

$\square$

Proof of Spectral Thm:

$\mathbb{C}$-version: " T normal $\Rightarrow \exists$ O.N. eigenbasis ".

$\mathbb{F} = \mathbb{C} \Rightarrow$ char. poly. of T splits "automatically".

Schur's $\Rightarrow \exists$ O.N.B. β s.t.

$$[T]_\beta = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix} = \begin{array}{c} \begin{matrix} \overset{\text{e-vector}}{v_1} & \overset{\therefore\ \text{e-vector}}{v_2} & & \end{matrix} \\ \begin{pmatrix} \lambda_1 & A_{12} & \cdots & A_{1n} \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \\ 0 & \vdots & 0 & \lambda_n \end{pmatrix} \end{array}$$

Claim: Indeed, it is diagonal. (by normality)

Pf: Want $A_{ij} = 0$ for $i < j$. $\beta = \{v_1, \dots, v_n\}$

$$\underset{\text{O.N.B. } \beta}{A_{ij}} = \langle T v_j, v_i \rangle = \langle v_j, T^* v_i \rangle \underset{T \text{ normal}}{=} \langle v_j, \bar{\lambda}_i v_i \rangle = 0$$

Repeat row by row. $\square$

$\mathbb{R}$-version: T self-adjoint $\Rightarrow$ T normal

Take any O.N.B. β for V,

then $A = [T]_\beta$ symmetric $n \times n$ $\mathbb{R}$ matrix.

Since $M_{n\times n}(\mathbb{R}) \subseteq M_{n \times n}(\mathbb{C})$,

so char. poly of A splits over $\mathbb{C}$

$\Rightarrow$ " " " " " " $\mathbb{R}$ (since T self-adj.)

$\overset{\text{Schur's}}{\Longrightarrow}$ $\exists$ β' O.N.B s.t

$$[T]_{\beta'} = \underbrace{\begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}}_{\substack{\text{symmetric} \\ + \\ \text{upper} \\ \text{triangular}}} \in M_{n\times n}(\mathbb{R})$$

$= \text{diagonal.}$

$\square$